Research Paper

Analytical solution for one-dimensional vertical electro-osmotic drainage under unsaturated conditions

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ARTICLE INFO

Keywords:
Electro-osmosis
Drainage
Unsaturated soil
Analytical solution

ABSTRACT

In this study, exponential functions were incorporated in a one-dimensional model to represent the relationships of hydraulic and electro-osmotic conductivities with matric suction and the soil-water characteristic curve. Analytical solutions for pore water pressure, water content and drainage were derived, and laboratory tests were performed to verify the effectiveness of these solutions. Finally, a parametric study indicated that the soil-water characteristics of the soil had a remarkable impact on the drainage behaviour of electro-osmosis under unsaturated conditions and that the proposed analytical solutions could be used to design electro-osmotic drainage systems in unsaturated soils.

1. Introduction

Electro-osmosis is an electrically induced process in which pore water is driven from the anode to the cathode through the dissolved electrolytes, thus leading to the dewatering and consolidation of saturated soil or other porous media. Casagrande [1] used this phenomenon to consolidate soft soil and enhance its geotechnical properties for engineering purposes. Since then, electro-osmosis has been successfully used for soft ground improvement, slope stabilisation, dewatering of tailings and sludge, pile foundation and soil remediation [2–12]. However, problems such as electrode erosion, low efficiency, uneven consolidation and unsatisfactory treatment results widely exist in practical applications, and improvement methods such as polarity reversal, application of alternating current, injection of a chemical solution and combination with other consolidation methods are highly desired by geotechnical engineers [13]. Recently, electro-osmosis has been used in unsaturated soils to enhance the performance of compacted clay liners at polluted sites by electrokinetic barriers [14–16], improve the stability of partially saturated slopes [17–18], reduce the adhesion of excavated clay materials on steel surfaces of tunnel driving machines [19] and decrease the water content of over-wet subgrade fill or expansive soil [20–23].

On the basis of the assumption that the pore water flow resulting from the hydraulic gradient and the electrical gradient can be superimposed linearly, the governing equation for electro-osmotic consolidation was developed, and many analytical solutions were derived using different conditions to analyse the development of pore water pressure and the degree of consolidation [24–31]. Esrig [24] first proposed a one-dimensional theoretical model and derived the solution for the excess pore water pressure under different boundary conditions. Wan and Mitchell [25] investigated the coupling effect of surcharge preloading and electro-osmotic consolidation. Following their pioneering work, several analytical models were proposed for electro-osmotic consolidation, including the two-dimensional model in a vertical plane [26–27] and the two-dimensional model in a horizontal plane [28]. Considering that prefabricated vertical drain and electric vertical drain are often installed in an equilateral triangular pattern in the ground, axisymmetric models were developed and the corresponding analytical solutions were derived [11,29]. Recently, the non-linear variations in soil compressibility and hydraulic and electro-osmotic conductivities were incorporated into a one-dimensional model, and analytical solutions for excess pore water pressure and degree of consolidation were derived [30,31]. These mathematical analyses generated significant knowledge pertaining to electro-osmotic consolidation and provided useful formulas for engineering design. However, all these analytical solutions were derived based on the assumption that the soil is fully saturated. The flow of pore water from the anode to the cathode changes the soil from a saturated state to an unsaturated state during electro-osmosis, thus leading to a decrease in the degree of saturation and non-linear variations in soil properties such as hydraulic

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https://doi.org/10.1016/j.compgeo.2018.09.011
Received 22 June 2018; Received in revised form 14 September 2018; Accepted 15 September 2018
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conductivity and electro-osmotic conductivity [32-37]. Such variations inevitably affect the development of pore water pressure and drainage during electro-osmosis, and predictions from the existing analytical solutions with the assumption of fully saturated soil would be inaccurate. However, to the best of our knowledge, there is no existing analytical solution for electro-osmotic drainage under unsaturated conditions.

In this study, a one-dimensional model for electro-osmotic drainage in the vertical direction considering unsaturated effects was proposed on the basis of the conservation of fluid mass, Darcy’s law and the electro-osmotic flow equation. Then, exponential functions were used to represent the relationships of hydraulic and electro-osmotic conductivities with matric suction and the soil-water characteristic curve, and these were incorporated into the one-dimensional model. The analytical solutions for pore water pressure, water content and drainage were derived, which were reasonably well comparable with the experimental results. Finally, a parametric study was conducted to investigate the influence of the soil-water characteristics on the drainage behaviour of electro-osmotic under unsaturated conditions.

2. Theoretical analysis

Similar to previous studies, a schematic diagram of a one-dimensional model for electro-osmotic drainage was developed in this study as shown in Fig. 1, with the anode on the bottom and the cathode on the top [30,31]. The bottom boundary is impermeable, and the top boundary is permeable. As in previous studies, the z axis was directed downwards [26,31]. Thus, the coordinate of any point in this one-dimensional model can be represented using a positive z value, which is consistent with the depth of the model, H. The following assumptions were made to develop the analytical model for electro-osmotic drainage under unsaturated conditions.

1. The soil is homogeneous, and under unsaturated conditions, the electrical properties of the soil mass are constant over time.
2. The pore water and soil grain are incompressible, the deformation of the soil is neglected and the drainage of pore water occurs in the vertical direction.
3. The velocity of pore water flow due to electro-osmosis is directly proportional to the electrical gradient and can be superimposed linearly due to the hydraulic gradient.
4. The pore water flow caused by the thermal gradient and chemical concentration gradient is neglected.
5. The differential equation describing electro-osmotic flow in unsaturated soil obeys the Richards’ equation.
6. The pore air pressure in the soil is kept unchanged, and a constant atmospheric pressure is maintained.

For the configuration shown in Fig. 1, a voltage \( V = V_0 \) is applied between the electrodes. In this case, the electric field intensity is expressed as follows:

\[
E = -\nabla V = \frac{\partial V}{\partial z} - i
\]

where \( E \) is the intensity of the electric field (in V/m) and \( i \) is a unit vector in the z direction. The negative sign in Eq. (1) denotes that the direction of the field intensity vector is from the anode (positive electrode) to the cathode (negative electrode). From assumption (1) that the electrical properties of the soil are homogeneous and constant over time, the electric field can be approximated as uniform within the soil mass. Thus, the distribution of the electrical voltage gradient between electrodes is assumed to be linear during the electrokinetic process. This assumption is in agreement with theory and the experimental observation at lower electric field intensities (typically, \( |E| < 100 \text{ V/m} \)) [26]. In addition, considering the voltage loss at the electrode-soil contact, the magnitude of the electric field intensity is usually lower than the total applied voltage divided by the electrode spacing, \( V_0/H \) (\( H \) is the height of the model), which has been observed in many laboratory and field tests [39].

By using the electric field intensity expressed in Eq. (1) and on the basis of the proportionality between the electrically induced velocity of water flow through the soil and the voltage gradient, the combined pore water flow induced by the hydraulic and electrical gradients during electro-osmosis can be described as follows [24]:

\[
v_s = -k_w \frac{\partial (u-u_s) + \frac{\partial V}{\partial z}}{\gamma_w} - k_{eo} \frac{\partial V}{\partial z}
\]

where \( v_s \) is the pore water flow in the vertical direction; \( \gamma_w \) is the unit weight of water; \( u \) is the pore water pressure (in such unsaturated situation, the matric suction is equal to the negative pore water pressure); and \( k_w \) and \( k_{eo} \) are the hydraulic and electro-osmotic conductivities of unsaturated soil, respectively.

On the basis of the conservation of fluid mass and assumptions (3) and (5) and neglecting the deformation of unsaturated soil, the one-dimensional non-linear differential equation that describes water flow in unsaturated soil during electro-osmotic drainage is expressed as follows:

\[
\frac{\partial}{\partial z} \left[ k_w \frac{\partial (u-u_s) + \frac{\partial V}{\partial z}}{\gamma_w} + k_{eo} \frac{\partial V}{\partial z} \right] = \frac{\partial \theta (z, t)}{\partial t} = \frac{\partial \theta}{\partial t} \frac{\partial u}{\partial t}
\]

where \( \theta \) is the volumetric water content, \( \partial \theta/\partial u \) is the storativity and \( t \) is the time.

Both the hydraulic conductivity and the volumetric water content of unsaturated soil are assumed to be functions of the matric suction [38]

\[
k_w (u) = k_r e^{\alpha u}
\]

\[
\partial (u) = \partial u + (\partial_1 - \partial) e^{\alpha u}
\]

where \( \theta_0 \) and \( \theta_e \) are the volumetric water content at saturation and the residual volumetric water content, respectively; \( k_r \) is the hydraulic conductivity at saturation and \( \alpha \) is the desaturation coefficient that represents the pore size distribution.

Similar to hydraulic conductivity under unsaturation conditions, electro-osmotic conductivity can also be defined as the product of the intrinsic conductivity, depending on the microstructural properties of the porous medium, and a non-dimensional relative conductivity coefficient, which accounts for the effects of partial saturation on soil electro-osmotic conduction properties. According to the experimental results obtained in electrokinetic filtration tests performed on unsaturated specimens [32,33], the exponential relationship between the electro-osmotic conductivity and the matric suction is also satisfied:

\[
k_{eo} (u) = k_c e^{\beta u}
\]

where \( k_c \) is the electro-osmotic conductivity at saturation and \( \beta \) is the relative electro-osmotic conductivity exponent. To facilitate the development of analytical solutions, the relative electro-osmotic conductivity exponent \( \beta \) is assumed to be equal to the desaturation coefficient \( \alpha \) in
By substituting the obtained pore water pressure into the soil-water characteristic curve (expressed as shown in Eq. (3b), the volumetric water content can be calculated, and then, the quantity of drainage at any time, Q(t), can be obtained as follows:

\[ Q = \int_{0}^{t} \theta_{0} \, dz - \int_{0}^{t} [\theta_{0} + (\theta_{0} - \theta_{e}) \, e^{(\theta_{0} - \theta_{e})}] \, dz \]

where \( \theta_{0} \) is the initial volumetric water content.

The integral in Eq. (17) cannot be expressed directly by elementary functions. Therefore, the Newton-Cotes formula is used to estimate its value.

### 3. Experimental validation

Laboratory tests and large-scale experiments on electro-osmotic drainage in unsaturated soil were conducted by Li and Zhuang et al. [40,41], respectively. However, these experiments were carried out to simulate two-dimensional electro-osmotic drainage, and the corresponding calculation parameters were not tested. Hence, it is difficult to adopt these experimental results to validate the proposed analytical solutions. In this study, a laboratory test was conducted to verify the capacity and calculation accuracy of a one-dimensional analytical solution for electro-osmotic drainage under unsaturated conditions. Soft soil used in the test was taken from Jiangxin Island, which is located in Nanjing, Jiangsu Province, China. Before the model test, the geotechnical properties of the soft soil and corresponding parameters included in the analytical solutions were tested, as listed in Table 1. Then, the soil was oven dried, mixed with water at a water content of 40% and compacted into the test device in five layers according to the predetermined void ratio in the range of 0.95–1.10. Finally, a plastic membrane was laid on the surface of the soil sample to prevent the evaporation of pore water.

As shown in Fig. 2, the electro-osmotic drainage test was conducted using a self-designed apparatus that was 200 mm in diameter and 550 mm in height. An aluminium sheet was used as the electrode, with the anode plate placed on the top of the soil sample and the porous membrane was laid on the surface of the soil sample to prevent the evaporation of pore water into a bottle placed on the electronic balance. To investigate the changes in the water content at different places between the anode and the cathode during electro-osmosis, four soil moisture sensors were installed along the height of the soil sample at intervals of 150 mm. The soil moisture sensor has a measurement range of 100% and a measuring accuracy of ±3%, with an output capacity of 2 V and 20 mA. In addition, the other devices included a direct current (DC) power supply with an output capacity of 60 V and 5 A, electrical wiring and five potential needles. The needles were Φ2 mm wires that were insulated, except the outermost 5 mm of the wire, positioned at 150 or 190 mm intervals. Needles E1 and E4 were inserted into the soil close to the anode and the cathode, respectively, which can measure the effective voltage applied to the soil sample.

When the sample preparation was completed, the drainage valve at the bottom of the test device was closed, and the whole system was...
allowed to stand for at least 2 days to reach equilibrium. Next, a DC voltage of 25 V was applied continuously for 172 h. In the electro-osmotic drainage test, the drainage, water content and voltage were monitored every 1–2 h and sometimes at long intervals such as at night under restricted conditions.

During the test, the effective voltage applied to the soil sample (between E1 and E4) was found to decrease with time because of the increase in the interfacial resistance with the duration of electro-osmosis. Considering the increase in voltage loss with time, an average effective voltage of 14 V was substituted into the analytical solutions. Except the effective voltage, the other corresponding parameters used in the analytical solutions are listed in Table 1, and the initial volumetric water content ($\theta_i$) along the height of soil sample was assumed to be uniform with a value of 0.5. Because the drainage direction in the test was opposite to that in the theoretical model in Fig. 1, Eq. (7b) can be rewritten as follows:

$$R_o = \frac{k_v}{k_s} E \gamma_o + \gamma_o$$

(18)

**Fig. 3** presents the time behaviour of the electro-osmotic drainage between the measured and the calculated values. The calculated result showed a good agreement with the experimental observation, although there was a little difference between the measured and calculated results during the initial 80 h. The faster increase in the measured drainage in the early stages can be attributed to the actual applied effective voltage in this period being greater than the actual applied effective voltage of the average effective voltage used in the analytical solutions. **Fig. 4** shows the comparisons of the volumetric water content at M1, M2 and M4 between the measured and the calculated results. Because of the failure of the soil moisture sensor of M3, the data for the volumetric water content at this point were not obtained. As shown in **Fig. 4**, the decrease in the volumetric water content for the soil around the anode was much larger than that near the cathode; this is consistent with the findings of previous experimental studies [42,43]. The volumetric water content for the soil near the cathode fluctuated throughout the test, although the overall trend showed a decrease. This phenomenon could be attributed to the formation of a relatively impermeable soil layer above the porous stone, which caused the pore water not to and accumulated near the cathode, which is different from the permeable boundary defined in the mathematical model. In general, the calculated volumetric water content is basically in good agreement with the observations, especially for soils around the anode (M1 and M2). Hence, this good agreement between the calculated and measured values indicates that the proposed analytical solutions can predict the behaviour of electro-osmotic drainage under unsaturated conditions.
4. Parametric study

To investigate the influence of the hydraulic conductivity and soil-water characteristic relationship on the time behaviour of the pore water pressure and drainage, a parametric study was conducted using the different values of the model parameters listed in Table 2. The height of the homogeneous soil was taken to be 1.0 m. The initial pore water pressure was assumed to be $u_i = 0$. The effective voltage was fixed at $V_0 = 50$ V. In the following sensitivity analysis, the influence of the ratio of the saturated electro-osmotic conductivity to the saturated hydraulic conductivity ($k_e/k_s$), desaturation coefficient ($\alpha$) and residual volumetric water content ($\theta_r$) were studied separately.

4.1. Effect of $k_e/k_s$

Fig. 5 shows the time variations in the average pore water pressures and drainage that correspond to the three values of $k_e/k_s$ (i.e., $k_e/k_s = 0.1$, $1.0$ and $10$), with other parameters fixed as follows: $k_s = 2.0 \times 10^{-9} \text{ m}^2 \text{s}^{-1} \text{V}^{-1}$, $\alpha = 0.01 \text{ kPa}^{-1}$, $\theta_s = 0.7$, and $\theta_r = 0.2$. As shown in Fig. 5(a), the value of $k_e/k_s$ had a significant influence on the development of the pore water pressure. The greater the value of $k_e/k_s$, the greater was the value of the matric suction. The drainage also increased with an increase in the value of $k_e/k_s$, as shown in Fig. 5(b). The variation in the drainage with $k_e/k_s$ was different from that in the matric suction. When $k_e/k_s$ increased from 0.1 to 1.0, the increase in the matric suction was much smaller than that when $k_e/k_s$ increased from 1.0 to 10. However, the increase in the drainage was much larger because of the exponential relationship between the volumetric water content and the matric suction (the soil-water characteristic curve). As the matric suction increased within a relatively small range, the volumetric water content decreased significantly. Nevertheless, when the matric suction was large and increased further, the volumetric water content of the soil decreased slightly and then tended to be invariable. Therefore, the drainage could be found to increase non-linearly with the increase in $k_e/k_s$, especially for soil with a large $\alpha$ value. In this situation, because electro-osmotic consolidation time would be prolonged with the increase in $k_e/k_s$, the determination of a suitable value of $k_e/k_s$ for electro-osmosis should consider the influence of the soil-water characteristic relationship.

4.2. Effect of $\alpha$

Three values of the desaturation coefficient ($\alpha = 0.005$, $0.01$ and $0.05 \text{ kPa}^{-1}$) were used to illustrate its effect on the time behaviour of pore water pressure and drainage under electro-osmosis. As shown in Fig. 6(a) and (b), the value of $k_e/k_s$ was taken as 1.0. Fig. 6(a) shows the variations in the average pore water pressure versus time for different values of $\alpha$.

Table 2

| $k_s$ ($1 \times 10^{-9} \text{m/s}$) | $k_e$ ($1 \times 10^{-9} \text{m}^2 \text{s}^{-1} \text{V}^{-1}$) | $\alpha$ ($\text{kPa}^{-1}$) | $\theta_s$ | $\theta_r$
|---|---|---|---|---|
| $0.2$, $2.0$ and $20$ | $2.0$ | $0.005$, $0.01$, $0.05$ | $0.7$ | $0.1$, $0.2$ and $0.3$

Fig. 5. Average pore water pressure and drainage versus time relationships for different $k_e/k_s$ ratios.

Fig. 6. Average pore water pressure and drainage versus time relationships for different values of $\alpha$.

$k_u$, the greater was the value of the matric suction. The drainage also increased with an increase in the value of $k_e/k_s$, as shown in Fig. 5(b). The variation in the drainage with $k_e/k_s$ was different from that in the matric suction. When $k_e/k_s$ increased from 0.1 to 1.0, the increase in the matric suction was much smaller than that when $k_e/k_s$ increased from 1.0 to 10. However, the increase in the drainage was much larger because of the exponential relationship between the volumetric water content and the matric suction (the soil-water characteristic curve). As the matric suction increased within a relatively small range, the volumetric water content decreased significantly. Nevertheless, when the matric suction was large and increased further, the volumetric water content of the soil decreased slightly and then tended to be invariable. Therefore, the drainage could be found to increase non-linearly with the increase in $k_e/k_s$, especially for soil with a large $\alpha$ value. In this situation, because electro-osmotic consolidation time would be prolonged with the increase in $k_e/k_s$, a larger value of $k_e/k_s$ is not suitable for electro-osmosis. Overall, the determination of a suitable value of $k_e/k_s$ for electro-osmosis should balance its effects on drainage volume and time and consider the influence of the soil-water characteristic relationship.
with \( \alpha = 0.005, 0.01 \) and \( 0.05 \) kPa\(^{-1}\). Because the changes in the hydraulic and electro-osmotic conductivities with the matric suction were assumed to be consistent in this study, the ratio of the hydraulic conductivity to electro-osmotic conductivity \( (k_{eo}/k_e) \) during electro-osmosis remained constant. Thus, the ultimate pore water pressures were equal for soil with different \( \alpha \) values. For the unsaturated soil, the desaturation coefficient is an important parameter related to the particle size distribution of the soil. A small value of \( \alpha \) leads to a high retention capability for water in the soil, which inhibits the drainage of the pore water. In contrast, a large value of \( \alpha \) means a low water retention capability, which promotes the drainage of the pore water. Therefore, the \( \alpha \) value has a significant effect on the variations of pore water pressure and drainage during electro-osmosis. The greater the \( \alpha \) value, the faster is the increase in the drainage and the slower is the development of the matric suction. Fig. 6(b) shows the influence of the desaturation coefficients on drainage under electro-osmosis. As expected, the drainage increased with the increase in \( \alpha \) value. As the \( \alpha \) value increased from 0.005 to 0.01 kPa\(^{-1}\), the drainage increased from 307.8 to 393.4 L/m\(^2\) (increase of 95.6 L/m\(^2\)). As the \( \alpha \) value increased further to 0.05 kPa\(^{-1}\), a further increase of 79.6 L/m\(^2\) was observed. Note that a non-linear relationship existed between the drainage and desaturation coefficient \( (\alpha) \), consistent with the soil-water characteristic curve, in which the relationship between the water content and the matric suction is described by an exponential function. Regarding the power of the exponential function, changes in the \( \alpha \) value within a relatively small range led to a considerable variation in the drainage. Therefore, considering that the main purpose of electro-osmotic drainage is to remove excessive water from unsaturated soils, it is preferable to have soil with a relatively large \( \alpha \) value.

4.3. Effect of \( \theta_r \)

The residual volumetric water content is also an important parameter in the soil-water characteristic curve, as shown in Eq. (4b). Three values of \( \theta_r \), 0.1, 0.2 and 0.3, were used to illustrate its effect while the other parameters remained the same. The value of \( k_{eo}/k_e \) was taken as 1.0, and the desaturation coefficient \( (\alpha) \) was taken as 0.01. Fig. 7(a) and (b) show the development of pore water pressure and drainage with time for soil with different \( \theta_r \) values. Together with the desaturation coefficient \( (\alpha) \), the residual volumetric water content \( (\theta_r) \) can reflect the water retention capability of the soil. The larger the value of \( \theta_r \), the higher is the water retention capability. Hence, for soil with a larger \( \theta_r \), the decrease in drainage is more remarkable, and the development of pore water pressure is faster. As Fig. 7(b) shows, the drainage decreased with the increase in \( \theta_r \), and the relationship between the drainage and \( \theta_r \) was linear. In addition, the drainage time (the time corresponding to 90% of final drainage) also decreased with the increase in \( \theta_r \). As the \( \theta_r \) value increased from 0.1 to 0.3, the drainage time decreased from 81 to 54 days. Hence, the effects of the residual volumetric water content \( (\theta_r) \) on the drainage volume and time should be balanced when designing an electro-osmotic drainage system in unsaturated soils.

5. Discussion

As mentioned above, the exponential coefficients for the hydraulic and electro-osmotic conductivities (\( \alpha \) and \( \beta \)) were assumed to be equal in this study to facilitate the development of analytical solutions. In this case, the effects of non-linear variations in \( k_{eo} \) and \( k_e \) were balanced by each other; therefore, the analytical solution for the ultimate pore water pressure simplifies to the equation proposed by Esrig [24] and Wan and Mitchell [25] as follows:

\[
\|_{ulim} = -\frac{k_{eo}}{k_e} V(z) + \|_{w} \tag{19}
\]

Thus, the use of the same exponential coefficients for the hydraulic and electro-osmotic conductivities will have a large effect on the prediction of the pore water pressure. To study the effect of this choice, numerical simulations were performed considering the non-synchronous variations in \( k_{eo} \) and \( k_w \) with the matric suction. The calculation model and parameters were the same as those in the above parametric study, with \( V_0 = 50 \) V, \( H = 1 \) m, \( k_e = 1 \times 10^{-5} \) m\(^2\) s\(^{-1}\) V\(^{-1}\), \( k_{eo}/k_e = 1 \), \( \theta_s = 0.7 \), \( \theta_r = 0.2 \), and \( \alpha = 0.01 \) kPa\(^{-1}\). According to the previous experimental studies by de Wet [32] and Gabrieli et al [33], with a decrease in the degree of saturation, the decrease in the electro-osmotic conductivity was found to be smaller than that in the hydraulic conductivity, which indicates \( \gamma > \beta \) in practice. Thus, in these simulations, the value of \( \beta \) was varied at 0.01, 0.009 and 0.008 kPa\(^{-1}\), while the value of \( \alpha \) was fixed at 0.01 kPa\(^{-1}\). Fig. 8(a) and (b) show the development of pore water pressure and drainage over time for soil with different \( \beta \) values. Fig. 8(a) shows that the difference in the pore water pressure was obvious for different \( \beta \) values, which means that the use of the same exponential coefficients (\( \beta = \alpha \)) may cause a large difference between the analytical solution and the experimental measurement of...
the pore water pressure. However, the difference in the drainage volumes for different $\beta$ values was minor, and the drainage volumes were approximately equal during the first 50 days, as shown in Fig. 8(b). At the final stage, the prediction errors in the drainage were only 2.1% and 4.1% in the cases of $\alpha = 0.01 \text{kPa}^{-1}$, $\beta = 0.008 \text{kPa}^{-1}$ and $\alpha = 0.01 \text{kPa}^{-1}$, $\beta = 0.009 \text{kPa}^{-1}$, respectively, based on the assumption of equal exponential coefficients. Fig. 9 shows the variations in the volumetric water content with depth and time for different $\beta$ values. The comparison result confirms that the assumption of equal exponential coefficients has less influence on the prediction of the water content. In sum, although a larger error will be caused by using the assumption of equal exponential coefficients to predict the pore water pressure, the predicted results of the water content and the quantity of drainage are acceptable. Thus, in this study, the analytical solutions developed on the basis of the assumption of equal exponential coefficients can reflect the behaviour of electro-osmotic drainage in unsaturated soils to some extent.

6. Summary and conclusions

In this study, a one-dimensional model for electro-osmotic drainage in the vertical direction under unsaturated conditions was proposed on the basis of the law of fluid mass conservation, Darcy’s law and the electro-osmotic flow equation. Exponential functions were used to represent the relationships of hydraulic and electro-osmotic conductivities with soil suction and the soil-water characteristic curve. A Kirchhoff variable and several dimensional variables were used to linearise the governing equation. Analytical solutions for the pore water pressure, water content and drainage were derived and compared with the results from a laboratory test for verification. Then, a parametric study was conducted to analyse the effect of the soil-water characteristic relationship on the behaviour of electro-osmotic drainage under unsaturated conditions.

The proposed analytical results agreed well with the experimental observations both for the water content and for the quantity of drainage; these results illustrate the accuracy of the proposed analytical solution. In addition, the calculated results demonstrated that the impact of the ratio of saturated electro-osmotic conductivity to saturated hydraulic conductivity ($k_e/k_s$), desaturation coefficient ($\alpha$) and residual volumetric water content ($\theta_r$) was significant on the development of the pore water pressure and drainage during electro-osmosis. More specifically, a larger ratio of the saturated electro-osmotic conductivity to saturated hydraulic conductivity ($k_e/k_s$) resulted in a higher pore water pressure. However, when the value of $k_e/k_s$ increased beyond a certain value, the drainage increased slowly, but the drainage time increased remarkably. Hence, the smaller the value of $k_e/k_s$, the higher the drainage. With the increase in the desaturation coefficient ($\alpha$) and the decrease in the residual volumetric water content ($\theta_r$), the matric suction developed more slowly, and the quantity of the drainage increased. Moreover, the drainage increased non-linearly with an increase in the $\alpha$ value because the soil-water characteristic curve satisfied the exponential relationship. According to this study, the influence of the water retention capability of soil is significant and should be considered in predictions of the drainage behaviour of electro-osmosis in unsaturated soils.

Acknowledgements

The authors greatly appreciate the guidance provided by Professor Caisheng Chen of Hohai University in the mathematical derivation. Financial support from the National Natural Science Foundation of China (Project No. 51509077) and the Fundamental Research Funds for the Central Universities (Project No. 2016B03514) are gratefully acknowledged. The authors also appreciate the reviewers’ excellent comments and suggestions, which helped to improve the quality of this paper.
Appendix 1

The governing equation is rearranged as follows:

$$\frac{\partial S}{\partial T} = \frac{\partial^2 S}{\partial Z^2}$$

(11)

The boundary conditions for the problem are

$$\begin{align*}
Z = 0, \ u = 0 & \Rightarrow S = e^{T/4} \\
Z = L, \ k_e(\bar{u}) \left( \frac{\partial u}{\partial x} \right)^2 + k_e(\bar{u}) & \Rightarrow \frac{\partial S}{\partial Z} + \frac{Z}{2} = 0
\end{align*}$$

(12)

and the initial condition can be set as

$$T = 0, \ u = u_0 \Rightarrow S = e^{u_0+Z/2}$$

(13)

We define new variables $V$ and $M$

\[ S(Z, T) = N(Z, T) + M(Z, T) \]

(S1)

that satisfy the following equations:

$$\begin{align*}
e^{T/4} &= S_{Z=0} = N_{Z=0} + M_{Z=0} \\
0 &= \frac{Z}{2} S_{Z=L} = \frac{N}{2} S_{Z=L} + \frac{1}{2} V_{Z=L} + \frac{1}{2} M_{Z=L}
\end{align*}$$

(S2)

Here, we let variable $M$ satisfy the following equations:

$$\begin{align*}
M_{Z=0} &= e^{T/4} \\
\frac{dM}{dZ}_{Z=L} &+ \frac{1}{2} M_{Z=L} = 0
\end{align*}$$

(S3)

We construct a linear function to satisfy Eq. (S3), which can be written as

$$M(Z, T) = e^{T/4} - \frac{Z}{2} + L$$

(S4)

Substituting Eqs. (S1) and (S4) into Eqs. (11) to (13), we have

$$\frac{\partial N}{\partial T} = \frac{\partial^2 N}{\partial Z^2} - \frac{\partial M}{\partial T}$$

(S5)

$$\begin{align*}
N_{T=0} &= Z_{T=0} - M_{T=0} = e^{u_0+Z/2} + \frac{Z}{2} - 1 \\
N_{Z=0} &= 0 \\
\frac{dN}{dZ}_{Z=L} &+ \frac{1}{2} V_{Z=L} = 0
\end{align*}$$

(S6)

Eq. (S5) is solved using the method of separation of variables with $N$ given by

$$N(Z, T) = P(Z)R(T)$$

(S7)

where $P$ is a function of $Z$ only and $R$ is a function of $T$ only; thus, we have the following eigenvalue equations:

$$\frac{d^2 P}{d Z^2} + \lambda P = 0$$

(S8)

$$\begin{align*}
P_{Z=0} &= 0 \\
\frac{dP}{dZ}_{Z=L} &+ \frac{1}{2} P_{Z=L} = 0
\end{align*}$$

(S9)

The solution of Eq. (S8) can be expressed as follows:

$$P(Z) = M_1 \cos(\beta Z) + M_2 \sin(\beta Z)$$

(S10)

where $M_1$, $M_2$ and $\beta$ are the integration constants from solving Eq. (S8). According to the boundary condition Eq. (S9), we have

$$\begin{align*}
\beta_n \cot(\beta_n L) &= -0.5, \ n = 1, 2, 3...
\end{align*}$$

(S11)

Substituting Eqs. (S13) and (S14) into Eq. (S12), the following equation can be obtained:

$$N(Z, T) = \sum_{n=1}^{\infty} R_n(T) \sin(\beta_n Z)$$

(S12)

Substituting Eq. (S12) into Eq. (S5) yields

$$\begin{align*}
\sum_{n=1}^{\infty} \left( \frac{dR_n}{dT} + \beta_n^2 R_n \right) \sin(\beta_n Z) &= -\frac{\partial M}{\partial T} = \left( \frac{Z}{8 + 4L} - \frac{1}{4} \right) e^{T/4}
\end{align*}$$

(S13)
Equation (S13) can be rewritten as follows:

\[
\frac{dR_n}{dt} + \beta_n^2 R_n \int_0^L \sin^2(\beta_n Z) dZ = \int_0^L \left( \frac{Z}{8 + 4L} - \frac{1}{4} \right) e^{\tau/4} \sin(\beta_n Z) dZ
\]  

(S14)

Here, we define

\[
A_n(\beta_n) = \int_0^L \sin^2(\beta_n Z) dZ
\]  

(S15)

Substituting Eq. (S15) into Eq. (S14), we have

\[
\frac{d(R_n e^{\beta_n^2 T})}{dT} = \frac{e^{\beta_n^2 T}}{A_n(\beta_n)} \int_0^L \left( \frac{Z}{8 + 4L} - \frac{1}{4} \right) e^{\tau/4} \sin(\beta_n Z) dZ
\]  

(S16)

Equation (S16) can be further rearranged as

\[
R_n(T) = R_n(0) e^{-\beta_n^2 T} + \frac{1}{A_n(\beta_n)} \int_0^T e^{\beta_n^2 (T-t)} \int_0^L \left( \frac{Z}{8 + 4L} - \frac{1}{4} \right) e^{\tau/4} \sin(\beta_n Z) dZ dt
\]  

(S16)

According to the initial condition in Eq. (S6) and Eq. (S12), we have

\[
e^{i \omega n Z/2} + \frac{Z}{2 + L} = 1 = N(Z, 0) = \sum_{n=1}^{\infty} R_n(0) e^{-\beta_n^2 Z}
\]  

(S17)

The following equation can be further obtained from Eq. (S17):

\[
\int_0^L \left( e^{i \omega n Z/2} + \frac{Z}{2 + L} \right) \sin(\beta_n Z) dZ = R_n(0) \int_0^L \sin^2(\beta_n Z) dZ = R_n(0) A_n(\beta_n)
\]  

(S18)

Here, we define

\[
B_n(\beta_n) = R_n(0) = \frac{1}{A_n(\beta_n)} \int_0^L \left( e^{i \omega n Z/2} + \frac{Z}{2 + L} \right) \sin(\beta_n Z) dZ
\]  

(S19)

Then, we introduce a new variable to simplify the solution,

\[
C_n(\beta_n, T) = R_n(T) = B_n(\beta_n) e^{\beta_n^2 T} + \frac{1}{A_n(\beta_n)} \int_0^T e^{\beta_n^2 (T-t)} \int_0^L \left( \frac{Z}{8 + 4L} - \frac{1}{4} \right) \sin(\beta_n Z) dZ dt
\]  

(S20)

Eq. (S12) can be written as

\[
N(Z, T) = \sum_{n=1}^{\infty} C_n(\beta_n, T) \sin(\beta_n Z)
\]  

(S21)

Substituting Eqs. (S4) and (S21) into Eq. (S11), we finally have

\[
S(Z, T) = \left( 1 - \frac{Z}{2 + L} \right) e^T + \sum_{n=1}^{\infty} C_n(\beta_n, T) \sin(\beta_n Z)
\]  

(S22)

Substituting Eq. (S22) into Eq. (11) yields

\[
u(z, t) = \frac{1}{a} \ln \left[ S(Z, T) - \frac{Z}{2 + T} - \frac{t}{4} \right]
\]  

(S23)

References


